SOLVING BOUNDARY-VALUE PROBLEMS IN HEAT CONDUCTION BY THE METHOD OF SUCCESSIVELY AVERAGING THE UNKNOWN FUNCTION

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An approximate analytical method is proposed by which linear boundary-value problems in heat conduction can be solved for an arbitrary distribution of heat sources and for boundary conditions of a general form.

Much attention is nowadays paid to the problem of finding simple and sufficiently accurate approximate solutions to boundary-value problems describing thermal fields in various structures.

For an approximate solution of problems in mathematical physics one uses variational methods [1-3], among which the Bubnov–Galerkin method ranks very high because of its simplicity and universality. Even more effective in many cases is the Kantorovich method [2] reduced to ordinary differential equations. These methods do, however, have also a number of drawbacks:

a) from the practical standpoint, the most serious drawback of these methods is the necessity of properly choosing the approximating (coordinate) functions which must satisfy either the boundary conditions (the Ritz method, the Bubnov–Galerkin method, the collocation method, the method of least squares) or the differential equation of the boundary-value problem (the Treffts method, the collocation method, the method of least squares) and this is rather difficult;

b) the properties of the operator in a boundary-value problem are accounted for exactly by a finite number of constants in the sought solution and by the sum of the products of these constants with the coordinate functions. If one tries to improve the accuracy by increasing the number of terms in the approximating expression, this will also result in much more complicated calculation formulas and will often actually worsen the accuracy on account of the accumulated computation error; therefore, this device is not always effective.

The method developed by Kantorovich partly overcomes these drawbacks, because here one considers the properties of the operator on a single independent variable only and its dependence on the other variables one stipulates a priori, which nevertheless improves the accuracy appreciably [2].

The authors propose an approximate analytical method for solving linear boundary-value problems which is entirely free of these drawbacks: it does not require matching the coordinate functions and, in a sense, takes into account the properties of the operator on all variables.

The basic concept of this method will be illustrated in the problem of finding the steady-state temperature field inside a homogeneous anisotropic parallelepiped $[-l_i \le \bar{i} \le l_i,\ \bar{i} = \bar{x}, \bar{y}, \bar{z}]$ with a stepped and symmetrical distribution of heat sources (Fig. 1), assuming that the heat transfer from it to the ambient medium follows Newton's law.

If we consider the symmetry of the problem, the latter will be reduced to finding the function which satisfies the equation of heat conduction:

$$\varepsilon_x \frac{\partial^2 N}{\partial x^2} + \varepsilon_y \frac{\partial^2 N}{\partial y^2} + \varepsilon_z \frac{\partial^2 N}{\partial z^2} = -f(x), \tag{1}$$

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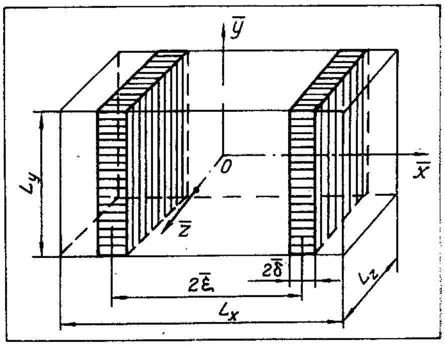

Fig. 1. Parallelepiped with stepped heat sources.

$$f(x)=\begin{cases}1 & \text{for } \xi-\delta\leqslant x\leqslant \xi+\delta,\\ 0 & \text{elsewhere, inside the cube for } [0\leqslant i\leqslant 1,\ i=x,\ y,\ z]\end{cases}$$

and the constraints

$$\left[\frac{\partial N}{\partial i}+B_iN\right]_{i=1}=0,\qquad \left[\frac{\partial N}{\partial i}\right]_{i=0}=0 \tag{2}$$

at its boundaries.

The following dimensionless variables and parameters have been introduced here:

$$i=\frac{2\bar{i}}{L_i},\quad B_i=\frac{\alpha_iL_i}{2\lambda_i},\quad \varepsilon_i=\frac{\lambda_i}{\lambda_m}\left(\frac{L_m}{L_i}\right)^2,$$

$$\xi=\frac{2\bar{\xi}}{L_x},\quad \delta=\frac{2\bar{\delta}}{L_x},\quad N=\frac{4\lambda_m(t-t_a)}{qvL_m^2}.$$

We proceed to solve the boundary-value problem (1)-(2). In the first stage we average the unknown function in the Oyz plane, for which we apply the operator I_{yz},

$$I_{yz}[N]=\int_0^1\int_0^1 N(x,\ y,\ z)\,dydz=N_x(x)=N_x$$

to the differential equation (1) with the constraints at boundaries x = 0, 1.

In view of the linearity of operator I_{yz}, we apply it term by term:

$$I_{yz}\left[\varepsilon_x\frac{\partial^2N}{\partial x^2}\right]=\varepsilon_x\frac{\partial^2}{\partial x^2}\left[\int_0^1\int_0^1 Ndydz\right]=\varepsilon_x\frac{d^2N_x}{dx^2};$$

$$I_{yz}\left[\varepsilon_y\frac{\partial^2N}{\partial y^2}\right]=\varepsilon_y\int_0^1 dz\int_0^1\frac{\partial^2N}{\partial y^2}dy=\varepsilon_y\int_0^1\left[\left.\frac{\partial N}{\partial y}\right|_{y=1}-\left.\frac{\partial N}{\partial y}\right|_{y=0}\right]dz.$$

According to the boundary conditions (2),

$$\left[\left.\frac{\partial N}{\partial y}\right|_{y=1}-\left.\frac{\partial N}{\partial y}\right|_{y=0}\right]=-B_yN(x,\ y=1,\ z).$$

Consequently,

$$I_{zy}\left[\varepsilon_y\frac{\partial^2N}{\partial y^2}\right]=-\varepsilon_yB_y\int_0^1 N(x,\ y=1,\ z)\,dz=-\varepsilon_yB_y\frac{\int_0^1 N(x,\ y=1,\ z)\,dz}{N_x}N_x=-\varepsilon_yB_y\psi_yN_x$$

with the coefficient

$$\psi_y=\frac{\int_0^1 N(x,\ y=1,\ z)\,dz}{N_x},$$

characterizing the nonuniformity of the temperature field at a section through abscissa x. Coefficient ψ_y is a function of x, but calculations for several specific cases have shown that this relation is weak and may be replaced – to the first approximation – by the ratio of average values of the respective functions, i.e.,

$$\psi_y\cong\tilde{\psi}_y=\frac{\int_0^1\int_0^1 N(x,\ y=1,\ z)\,dxdz}{\int N_xdx}=\frac{N_S(y=1)}{N_V}, \tag{3}$$

to be used in the subsequent analysis. Analogously,

$$I_{yz}\left[\varepsilon_z\frac{\partial^2N}{\partial z^2}\right]=-\varepsilon_zB_z\tilde{\psi}_zN_x,\quad \tilde{\psi}_z=\frac{N_S(z=1)}{N_V};$$

$$I_{yz}[f(x)]=f(x).$$

Adding the results of the I_{yz} operation to Eq. (1), we arrive at the following ordinary differential equation in N_x:

$$\frac{d^2N_x}{dx^2} - p_x^2 N_x = -\frac{1}{\varepsilon_x} f(x),$$
$$p_x^2 = \frac{\varepsilon_y B_y \tilde{\psi}_y + \varepsilon_z B_z \tilde{\psi}_z}{\varepsilon_x}. \tag{4}$$

Applying operator I_{yz} also to the constraints in (2) at x = 0, 1 we obtain the boundary conditions for function N_x:

$$\left[\frac{dN_x}{dx} + B_x N_x\right]_{x=1} = 0, \quad \left[\frac{dN_x}{dx}\right]_{x=0} = 0. \tag{5}$$

Integrating Eq. (4) and satisfying conditions (5) will yield

$$N_x = \frac{2}{\varepsilon_x p_x^2}(\varphi_{1x} - \varphi_{2x}),$$
$$\varphi_{1x} = A\frac{\operatorname{ch} p_x x}{\operatorname{ch} p_x}, \qquad \varphi_{2x} = \begin{cases} 0, & x \leqslant \xi - \delta, \\ \operatorname{sh}^2 p_x \dfrac{x - (\xi - \delta)}{2}, & \xi - \delta \leqslant x \leqslant \xi + \delta, \\ \operatorname{sh} p_x\delta \operatorname{sh} p_x (x - \xi), & x > \xi + \delta, \end{cases}$$
$$\Omega_x = \left(1 + \frac{p_x \operatorname{th} p_x}{B_x}\right)^{-1}, \tag{6}$$
$$A = \Omega_x \operatorname{sh} p_x \delta \operatorname{ch} p_x (1 - \xi)\left[\frac{p_x}{B_x} + \operatorname{th} p_x (1 - \xi)\right].$$

We now return to the original equation (1), rewriting it as

$$\varepsilon_y \frac{\partial^2 N}{\partial y^2} + \varepsilon_z \frac{\partial^2 N}{\partial z^2} = -f(x) - \varepsilon_x \frac{\partial^2 N}{\partial x^2}.$$

With the aid of Eq. (4) and approximating the last equation

$$\frac{\partial^2 N}{\partial x^2} \cong \frac{d^2 N_x}{dx^2} = p_x^2 N_x - \frac{1}{\varepsilon_x} f(x), \tag{7}$$

we obtain

$$\varepsilon_y \frac{\partial^2 M}{\partial y^2} + \varepsilon_z \frac{\partial^2 M}{\partial z^2} = -2(\varphi_{1x} - \varphi_{2x}), \tag{8}$$

where M(y, z; x) is the approximation to the unknown function N based on assumptions (3) and (7). Here x appears already as a parameter. At the boundaries, M must satisfy the original conditions (2)

$$\left[\frac{\partial M}{\partial i} + B_i M\right]_{i=1} = 0, \quad \left[\frac{\partial M}{\partial i}\right]_{i=0} = 0, \quad i = y,\ z. \tag{9}$$

In order to solve the boundary-value problem (8)-(9), we again use the same algorithm to reduce the problem to an ordinary differential equation. Applying operator I_z

$$I_z[M] = \int_0^1 M dz = M_y,$$

we arrive at the boundary-value problem with respect to M_y:

$$\frac{d^2 M_y}{dy^2} - p_y^2 M_y = -\frac{2}{\varepsilon_y}(\varphi_{1x} - \varphi_{2x}),$$
$$p_y^2 = \frac{\varepsilon_z \tilde{\psi}_z B_z}{\varepsilon_y}, \tag{11}$$
$$\left[\frac{dM_y}{dy} + B_y M_y\right]_{y=1} = 0, \quad \left[\frac{dM_y}{dy}\right]_{y=0} = 0,$$

and from here

$$M_y = \frac{2}{\varepsilon_y p_y^2}(\varphi_{1x} - \varphi_{1x})\varphi_y,$$

TABLE 1. Relative Errors ($\delta\tilde{N}$) of the First Approximations to N Obtained by the Averaging Method (O), by the Galerkin–Ritz Method (G–R) and also by the Kantorovich Method (K) for the Center Point of a Parallelepiped

L_z/L_y	0			0,2			0,5			1		
L_z/L_x	O	G–R	K	O	G–R	K	O	G–R	K	O	G–R	K
0	0,0	56,6	25,0	2,1	53,7	22,3	2,6	36,8	9,4	9,4	30,4	4,0
0,2	2,1	53,7	27,6	3,8	50,2	24,4	4,8	35,3	11,6	10,1	27,8	4,7
0,5	2,6	36,8	25,2	6,3	35,3	23,0	8,2	25,8	12,8	14,9	22,1	6,7
1	9,4	30,4	26,1	12,1	27,8	24,2	17,9	22,1	17,2	20,9	15,6	9,8

$$\varphi_y = 1 - \Omega_y \frac{\operatorname{ch} p_y y}{\operatorname{ch} p_y}, \quad \Omega_y = \left(1 + \frac{p_y \operatorname{th} p_y}{B_y}\right)^{-1}. \tag{12}$$

In the final stage of the solution we replace $\partial^2 M/\partial y^2$ in Eq. (8) by its approximation

$$\frac{d^2 M_y}{dy^2} = p_y^2 M_y - \frac{2}{\varepsilon_y}(\varphi_{1x} - \varphi_{2x}),$$

and, according to (10), reduce the original boundary-value problem (1)-(2) to a one-dimensional one:

$$\frac{d^2\tilde{N}}{dz^2} = -\frac{2}{\varepsilon_z}(\varphi_{1x} - \varphi_{2x})\varphi_y, \tag{13}$$

$$\left[\frac{d\tilde{N}}{dz} + B_z\tilde{N}\right]_{z=1} = 0, \quad \left[\frac{d\tilde{N}}{dz}\right]_{z=0} = 0, \tag{14}$$

where $\tilde{N}$ is the approximation for the unknown function based on type (3) and (7) assumptions, which have been made for all stages of the solution.

Integration of Eq. (13) with the corresponding boundary conditions (14) yields the approximate solution to the original boundary-value problem (1)-(2) in the form

$$\tilde{N} = \frac{1}{\varepsilon_z}(\varphi_{1x} - \varphi_{2x})\varphi_y\varphi_z, \quad \varphi_z = 1 + \frac{2}{B_z} - z^2. \tag{15}$$

In order to be able to use formula (15) for calculations, one must know the parameters p_x and p_y, which in turn are expressed through the nonuniformity factors $\tilde{\psi}_y$ and $\tilde{\psi}_z$. The latter will be determined from (3) and the approximate solution (15):

$$\tilde{\psi}_z = \frac{\tilde{N}_S(z=1)}{\tilde{N}_V} = \frac{1}{1 + \frac{B_z}{3}}, \quad \tilde{\psi}_y = \frac{\tilde{N}_S(y=1)}{N_V} = \frac{1}{1 + \frac{B_y}{m}},$$

$$m = \frac{p_y^2 \operatorname{th} p_y}{p_y - \operatorname{th} p_y}, \quad p_y^2 = \frac{\varepsilon_z B_z \tilde{\psi}_z}{\varepsilon_y}.$$

Let us examine the error of the approximate solution (15). Methods of estimating the error of approximate solutions to boundary-value problems [1] have still not been developed well enough to be applicable to other than the most simple cases, but for practical purposes we will assess the effectiveness of our method by comparing the approximate solutions with known exact solutions [1-3].

We will consider two special cases of the problem.

1. The heat source occupies the entire space inside the parallelepiped, i.e., $\xi = \delta = 0.5$. Inserting this value for ξ and δ into (15), we obtain an expression for the temperature field here:

$$\tilde{N} = \frac{1}{2\varepsilon_z}\varphi_x\varphi_y\varphi_z, \quad \varphi_x = 1 - \Omega_x \frac{\operatorname{ch} p_x x}{\operatorname{ch} p_x}. \tag{16}$$

It follows from physical considerations that the largest errors occur at $B_i = \infty$, i.e., in the boundary-value problem with constraints of the first kind and that, therefore, it makes sense to evaluate the error of solution (16) just for this extreme case.

TABLE 2. Relative Error ($\delta\tilde{N}$) of the Mean Value of N on the Surface of the Source

L_y/L_z	1	1,25	1,5	2	3	5	10	∞
$\frac{L_x}{L_z}=1$	10,0	8,3	6,9	5,5	4,5	4,2	5,0	4,8

Computation formulas are easily obtained for the general case (16) by going to the limit $B_i \to \infty$ and letting $\lambda_m = \lambda_z$, $L_m = L_z$.

Here $B_z\tilde{\psi}_z \to 3$, $B_y\tilde{\psi}_y \to m$, and $\Omega_i \to 1$. Consequently,

$$\tilde{N} = \frac{1}{2}\left(1 - \frac{\operatorname{ch} p_x x}{\operatorname{ch} p_x}\right)\left(1 - \frac{\operatorname{ch} p_y y}{\operatorname{ch} p_y}\right)(1 - z^2),$$
$$p_y^2 = \frac{3}{\varepsilon_y}, \quad m = \frac{p_y^2 \operatorname{th} p_y}{p_y - \operatorname{th} p_y}, \quad p_x^2 = \frac{m\varepsilon_y + 3}{\varepsilon_x}. \tag{17}$$

The exact solution to this particular problem is given in [4], and we will compare our approximate solution (17) with it. The results have been compiled in Table 1, with the relative errors of the approximate solution $\delta\tilde{N}$ given for the center point 0(0, 0, 0).

The first approximations obtained by the proposed method of averaging are compared also with those obtained by the Galerkin–Ritz and by the Kantorovich method. Table 1 indicates that averaging yields, as a rule, a better accuracy.

Calculations have shown that $m \approx 3$, which suggests the following universal approximation formula for ψ_i:

$$\tilde{\psi}_i = \frac{1}{1 + \frac{B_i}{3}}. \tag{18}$$

The most accurate approximate solution was obtained with this choice of values for the nonuniformity factor.

2. The heat source is located on the boundary x = 1, which is thermally insulated.

The temperature field in this case is highly nonuniform and can be described by expression (15) with $B_x \to 0$, $\xi = 1$, and $\delta \to 0$. Furthermore, considering that $q_V = Q/2\delta L_z L_y \to q_S = Q/L_z L_y$, $\tilde{N}$ must be additionally multiplied by $1/2\delta$. As a result of the limiting process at B_y, $B_z \to \infty$, we have

$$\tilde{N} = \frac{p_x}{2}\,\frac{\operatorname{ch} p_x x}{\operatorname{sh} p_x}\left(1 - \frac{\operatorname{ch} p_y y}{\operatorname{ch} p_y}\right)(1 - z^2),$$
$$p_y^2 = \frac{3}{\varepsilon_y}, \quad p_x^2 = \frac{3 + m\varepsilon_y}{\varepsilon_x}, \quad m = \frac{p_y^2 \operatorname{th} p_y}{p_y - \operatorname{th} p_y}, \quad \varepsilon_y = \left(\frac{L_z}{L_y}\right)^2, \quad \varepsilon_x = \left(\frac{L_z}{L_x}\right)^2. \tag{19}$$

The exact solution in [4] applies to the case where the heat source occupies one side, while in our symmetrical case it occupies two opposite sides, but for our case too the respective exact solution can be obtained by applying the superposition principle.

An important quantity characterizing the thermal resistance in our case is the mean-integral value of the unknown function N at the source surface [4]. The values of this function for various ratios of parallelepiped dimensions were determined from the approximate solution (19) and compared with its exact values. The relative errors are shown in Table 2. Evidently, this quantity is almost independent of the parameter L_z/L_x and, for this reason, results are shown for $L_z/L_x = 1$ only.

We note, in conclusion, that solution (15) must be treated as the first approximation.

NOTATION

L_i	is the parallelepiped dimension along the Oi axis;
l_i	is the half-dimension;
λ_i	is the thermal conductivity;
α_i	is the coefficient of heat transfer from the boundary i = 1;
Q	is the total power of heat sources;

q_V	is the volume density of heat sources;
q_S	is the surface density of heat sources;
t_a	is the ambient temperature;
t	is the body temperature;
B_i	is the Biot number;
L_m, λ_m	are the scale values of the respective quantities;
N	is the dimensionless temperature;
$N_S(i = 1)$	is the mean value of the function at the boundary $i = 1$;
N_V	is the mean-over-volume value of N;
$\delta\tilde{N} = ((\tilde{N}-N)/N)100\%$	is the relative error of the approximate solution.

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